NOTE ON MATH 2060: MATHEMATICAL ANALYSIS II: 2017-18

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1. Riemann Integrable Functions

We will use the following notation throughout this chapter.

(i): All functions $f, g, h...$ are bounded real valued functions defined on $[a, b]$ and $m \le f \le M$ on $[a, b]$.
(ii): Let $P : a = x_0 < x_1 < .... < x_n = b$ denote a partition on $[a, b]$; Put $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max \Delta x_i$.
(iii): $M_i(f, P) := \sup\{f(x) : x \in [x_{i-1}, x_i\}$; $m_i(f, P) := \inf\{f(x) : x \in [x_{i-1}, x_i\}$.
Set $\omega_i(f, P) = M_i(f, P) - m_i(f, P)$.
(iv): (the *upper sum* of f): $U(f, P) := \sum M_i(f, P)\Delta x_i$
(the *lower sum* of f). $L(f, P) := \sum m_i(f, P)\Delta x_i$.

Remark 1.1. *It is clear that for any partition on $[a, b]$, we always have*
(i) $m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$.
(ii) $L(-f, P) = -U(f, P)$ *and* $U(-f, P) = -L(f, P)$.

The following lemma is the critical step in this section.

Lemma 1.2. *Let P and Q be the partitions on $[a, b]$. We have the following assertions.*
(i) If $P \subseteq Q$, then $L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P)$.
(ii) We always have $L(f, P) \le U(f, Q)$.

Proof. For Part (i), we first claim that $L(f, P) \le L(f, Q)$ if $P \subseteq Q$. By using the induction on $l := \#Q - \#P$, it suffices to show that $L(f, P) \le L(f, Q)$ as $l = 1$. Let $P : a = x_0 < x_1 < \cdots < x_n = b$ and $Q = P \cup \{c\}$. Then $c \in (x_{s-1}, x_s)$ for some s. Notice that we have

$$m_s(f, P) \le \min\{m_s(f, Q), m_{s+1}(f, Q)\}.$$

So, we have

$$m_s(f, P)(x_s - x_{s-1}) \le m_s(f, Q)(c - x_{s-1}) + m_{s+1}(f, Q)(x_s - c).$$

This gives the following inequality as desired.

$$L(f, Q) - L(f, P) = m_s(f, Q)(c - x_{s-1}) + m_{s+1}(f, Q)(x_s - c) - m_s(f, P)(x_s - x_{s-1}) \ge 0. \quad (1.1)$$

Now by considering $-f$ in the Inequality 1.1 above, we see that $U(f, Q) \le U(f, P)$.
For Part (ii), let P and Q be any pair of partitions on $[a, b]$. Notice that $P \cup Q$ is also a partition on $[a, b]$ with $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$. So, Part (i) implies that

$$L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q).$$

The proof is complete. □

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The following plays an important role in this chapter.

Definition 1.3. *Let f be a bounded function on $[a,b]$. The upper integral (resp. lower integral) of f over $[a,b]$, write $\overline{\int_a^b} f$ (resp. $\underline{\int_a^b} f$), is defined by*

$$\overline{\int_a^b} f = \inf\{U(f,P) : P \text{ is a partation on } [a,b]\}.$$

(resp.

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is a partation on } [a,b]\}.)$$

Notice that the upper integral and lower integral of f must exist by Remark 1.1.

Proposition 1.4. *Let f and g both are bounded functions on $[a,b]$. With the notation as above, we always have*

(i)

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(ii) $\underline{\int_a^b}(-f) = -\overline{\int_a^b} f$.

(iii)

$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f+g) \leq \overline{\int_a^b} (f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g.$$

Proof. Part (i) follows from Lemma 1.2 at once.
Part (ii) is clearly obtained by $L(-f,P) = -U(f,P)$.
For proving the inequality $\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g) \leq$ first. It is clear that we have $L(f,P) + L(g,P) \leq L(f+g,P)$ for all partitions P on $[a,b]$. Now let P_1 and P_2 be any partition on $[a,b]$. Then by Lemma 1.2, we have

$$L(f,P_1) + L(g,P_2) \leq L(f,P_1 \cup P_2) + L(g,P_1 \cup P_2) \leq L(f+g,P_1 \cup P_2) \leq \underline{\int_a^b}(f+g).$$

So, we have

$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f+g). \tag{1.2}$$

As before, we consider $-f$ and $-g$ in the Inequality 1.2, we get $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$ as desired. □

The following example shows the strict inequality in Proposition 1.4 (iii) may hold in general.

Example 1.5. *Define a function $f, g : [0,1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1 & \textit{if } \; x \in [0,1] \cap \mathbb{Q}; \\ -1 & \textit{otherwise.} \end{cases}$$

and

$$g(x) = \begin{cases} -1 & \text{if } x \in [0,1] \cap \mathbb{Q}; \\ 1 & \text{otherwise.} \end{cases}$$

Then it is easy to see that $f + g \equiv 0$ and

$$\overline{\int_0^1} f = \overline{\int_0^1} g = 1 \quad \text{and} \quad \underline{\int_0^1} f = \underline{\int_0^1} g = -1.$$

So, we have

$$-2 = \underline{\int_a^b} f + \underline{\int_a^b} g < \underline{\int_a^b} (f+g) = 0 = \overline{\int_a^b} (f+g) < \overline{\int_a^b} f + \overline{\int_a^b} g = 2.$$

We can now reaching the main definition in this chapter.

Definition 1.6. *Let f be a bounded function on $[a,b]$. We say that f is Riemann integrable over $[a,b]$ if $\overline{\int_b^a} f = \underline{\int_a^b} f$. In this case, we write $\int_a^b f$ for this common value and it is called the Riemann integral of f over $[a,b]$.*
Also, write $R[a,b]$ for the class of Riemann integrable functions on $[a,b]$.

Proposition 1.7. *With the notation as above, $R[a,b]$ is a vector space over $\mathbb{R}$ and the integral*

$$\int_a^b : f \in R[a,b] \mapsto \int_a^b f \in \mathbb{R}$$

defines a linear functional, that is, $\alpha f + \beta g \in R[a,b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ for all $f, g \in R[a,b]$ and $\alpha, \beta \in \mathbb{R}$.

Proof. Let $f, g \in R[a,b]$ and $\alpha, \beta \in \mathbb{R}$. Notice that if $\alpha \geq 0$, it is clear that $\overline{\int_a^b} \alpha f = \alpha \overline{\int_a^b} f = \alpha \int_a^b f = \alpha \underline{\int_a^b} f = \underline{\int_a^b} \alpha f$. Also, if $\alpha < 0$, we have $\overline{\int_a^b} \alpha f = \alpha \underline{\int_a^b} f = \alpha \int_a^b f = \alpha \overline{\int_a^b} f = \underline{\int_a^b} \alpha f$. Therefore, we have $\int_a^b \alpha f = \alpha \int_a^b f$ for all $\alpha \in \mathbb{R}$. For showing $f + g \in R[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, these will follows from Proposition 1.4 (*iii*) at once. The proof is finished. □

The following result is the important characterization of a Riemann integrable function. Before showing this, we will use the following notation in the rest of this chapter.
For a partition $P : a = x_0 < x_1 < \cdots < x_n = b$ and $1 \leq i \leq n$, put

$$\omega_i(f, P) := \sup\{|f(x) - f(x')| : x, x' \in [x_{i-1}, x_i]\}.$$

It is easy to see that $U(f,P) - L(f,P) = \sum_{i=1}^n \omega_i(f,P) \Delta x_i$.

Theorem 1.8. *Let f be a bounded function on $[a,b]$. Then $f \in R[a,b]$ if and only if for all $\varepsilon > 0$, there is a partition $P : a = x_0 < \cdots < x_n = b$ on $[a,b]$ such that*

$$0 \leq U(f,P) - L(f,P) = \sum_{i=1}^n \omega_i(f,P) \Delta x_i < \varepsilon. \tag{1.3}$$

Proof. Suppose that $f \in R[a,b]$. Let $\varepsilon > 0$. Then by the definition of the upper integral and lower integral of f, we can find the partitions P and Q such that $U(f,P) < \overline{\int_a^b} f + \varepsilon$ and $\underline{\int_a^b} f - \varepsilon < L(f,Q)$. By considering the partition $P \cup Q$, we see that

$$\underline{\int_a^b} f - \varepsilon < L(f,Q) \leq L(f, P\cup Q) \leq U(f, P\cup Q) \leq U(f,P) < \overline{\int_a^b} f + \varepsilon.$$

Since $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$, we have $0 \leq U(f,P\cup Q) - L(f, P\cup Q) < 2\varepsilon$. So, the partition $P \cup Q$ is as desired.

Conversely, let $\varepsilon > 0$, assume that the Inequality 1.3 above holds for some partition P. Notice that we have

$$L(f,P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f,P).$$

So, we have $0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon$ for all $\varepsilon > 0$. The proof is finished. □

Remark 1.9. *Theorem 1.8 tells us that a bounded function f is Riemann integrable over $[a,b]$ if and only if the* "size" *of the discontinuous set of f is arbitrary small.*

Example 1.10. *Let $f : [0,1] \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} \frac{1}{p} & \textit{if } x = \frac{q}{p}, \textit{ where } p, q \textit{ are relatively prime positive integers}; \\ 0 & \textit{otherwise.} \end{cases}$$

Then $f \in R[0,1]$.

(Notice that the set of all discontinuous points of f, say D, is just the set of all $(0,1] \cap \mathbb{Q}$. Since the set $(0,1] \cap \mathbb{Q}$ is countable, we can write $(0,1] \cap \mathbb{Q} = \{z_1, z_2, ...\}$. So, if we let $m(D)$ be the "size" of the set D, then $m(D) = m(\bigcup_{i=1}^\infty \{z_i\}) = \sum_{i=1}^\infty m(\{z_i\}) = 0$, in here, you may think that the size of each set $\{z_i\}$ is 0.)

Proof. Let $\varepsilon > 0$. By Theorem 1.8, it aims to find a partition P on $[0,1]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Notice that for $x \in [0,1]$ such that $f(x) \geq \varepsilon$ if and only if $x = q/p$ for a pair of relatively prime positive integers p, q with $\frac{1}{p} \geq \varepsilon$. Since $1 \leq q \leq p$, there are only finitely many pairs of relatively prime positive integers p and q such that $f(\frac{q}{p}) \geq \varepsilon$. So, if we let $S := \{x \in [0,1] : f(x) \geq \varepsilon\}$, then S is a finite subset of $[0,1]$. Let L be the number of the elements in S. Then, for any partition $P : a = x_0 < \cdots < x_n = 1$, we have

$$\sum_{i=1}^n \omega_i(f,P)\Delta x_i = \Big(\sum_{i:[x_{i-1},x_i]\cap S = \emptyset} + \sum_{i:[x_{i-1},x_i]\cap S \neq \emptyset} \Big)\, \omega_i(f,P)\Delta x_i.$$

Notice that if $[x_{i-1}, x_i] \cap S = \emptyset$, then we have $\omega_i(f,P) \leq \varepsilon$ and thus,

$$\sum_{i:[x_{i-1},x_i]\cap S = \emptyset} \omega_i(f,P)\Delta x_i \leq \varepsilon \sum_{i:[x_{i-1},x_i]\cap S = \emptyset} \Delta x_i \leq \varepsilon(1-0).$$

On the other hand, since there are at most $2L$ sub-intervals $[x_{i-1}, x_i]$ such that $[x_{i-1}, x_i] \cap S \neq \emptyset$ and $\omega_i(f,P) \leq 1$ for all $i = 1, ..., n$, so, we have

$$\sum_{i:[x_{i-1},x_i]\cap S \neq \emptyset} \omega_i(f,P)\Delta x_i \leq 1 \cdot \sum_{i:[x_{i-1},x_i]\cap S \neq \emptyset} \Delta x_i \leq 2L\|P\|.$$

We can now conclude that for any partition P, we have

$$\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i \le \varepsilon + 2L\|P\|.$$

So, if we take a partition P with $\|P\| < \varepsilon/(2L)$, then we have $\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i \le 2\varepsilon$.
The proof is finished. □

Proposition 1.11. *Let f be a function defined on $[a,b]$. If f is either monotone or continuous on $[a,b]$, then $f \in R[a,b]$.*

Proof. We first show the case of f being monotone. We may assume that f is monotone increasing. Notice that for any partition $P : a = x_0 < \cdots < x_n = b$, we have $\omega_i(f,P) = f(x_i) - f(x_{i-1})$. So, if $\|P\| < \varepsilon$, we have

$$\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i = \sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))\Delta x_i < \|P\| \sum_{i=1}^{n}(f(x_i)-f(x_{i-1})) = \|P\|(f(b)-f(a)) < \varepsilon(f(b)-f(a)).$$

Therefore, $f \in R[a,b]$ if f is monotone.
Suppose that f is continuous on $[a,b]$. Then f is uniform continuous on $[a,b]$. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ as $x, x' \in [a,b]$ with $|x - x'| < \delta$. So, if we choose a partition P with $\|P\| < \delta$, then $\omega_i(f,P) < \varepsilon$ for all i. This implies that

$$\sum_{i=1}^{n} \omega_i(f,P)\Delta x_i \le \varepsilon \sum_{i=1}^{n} \Delta x_i = \varepsilon(b-a).$$

The proof is complete. □

Proposition 1.12. *We have the following assertions.*

(i) If $f, g \in R[a,b]$ with $f \le g$, then $\int_a^b f \le \int_a^b g$.

(ii) If $f \in R[a,b]$, then the absolute valued function $|f| \in R[a,b]$. In this case, we have $|\int_a^b f| \le \int_a^b |f|$.

Proof. For Part (i), it is clear that we have the inequality $U(f,P) \le U(g,P)$ for any partition P. So, we have $\int_a^b f = \overline{\int_a^b} f \le \overline{\int_a^b} g = \int_a^b g$.
For Part (ii), the integrability of $|f|$ follows immediately from Theorem 1.8 and the simple inequality $||f|(x') - |f|(x'')| \le |f(x') - f(x'')|$ for all $x', x'' \in [a,b]$. Thus, we have $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$ for any partition P on $[a,b]$.
Finally, since we have $-f \le |f| \le f$, by Part (i), we have $|\int_a^b f| \le \int_a^b |f|$ at once. □

Proposition 1.13. *Let $a < c < b$. We have $f \in R[a,b]$ if and only if the restrictions $f|_{[a,c]} \in R[a,c]$ and $f|_{[c,b]} \in R[c,b]$. In this case we have*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{1.4}$$

Proof. Let $f_1 := f|_{[a,c]}$ and $f_2 := f|_{[c,b]}$.
It is clear that we always have

$$U(f_1,P_1) - L(f_1,P_1) + U(f_2,P_2) - L(f_2,P_2) = U(P,f) - L(f,P)$$

for any partition P_1 on $[a,c]$ and P_2 on $[c,b]$ with $P = P_1 \cup P_2$.
From this, we can show the sufficient condition at once.
For showing the necessary condition, since $f \in R[a,b]$, for any $\varepsilon > 0$, there is a partition Q on $[a,b]$

such that $U(f,Q) - L(f,Q) < \varepsilon$ by Theorem 1.8. Notice that there are partitions P_1 and P_2 on $[a,c]$ and $[c,b]$ respectively such that $P := Q \cup \{c\} = P_1 \cup P_2$. Thus, we have

$$U(f_1,P_1) - L(f_1,P_1) + U(f_2,P_2) - L(f_2,P_2) = U(f,P) - L(f,P) \leq U(f,Q) - L(f,Q) < \varepsilon.$$

So, we have $f_1 \in R[a,c]$ and $f_2 \in R[c,b]$.
It remains to show the Equation 1.4 above. Notice that for any partition P_1 on $[a,c]$ and P_2 on $[c,b]$, we have

$$L(f_1,P_1) + L(f_2,P_2) = L(f,P_1 \cup P_2) \leq \underline{\int_a^b} f = \int_a^b f.$$

So, we have $\int_a^c f + \int_c^b f \leq \int_a^b f$. Then the inverse inequality can be obtained at once by considering the function $-f$. Then the resulted is obtained by using Theorem 1.8. □